

1. Numerical Analysis of RC and RL Circuits with Cyclic Parameter Variation

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Series RC and RL circuits driven by an AC voltage source have been analysed using numerical methods including the effects of sinusoidal variation of inductance and capacitance with respect to time and at frequencies which are equal to or are multiple of the AC voltage source frequency. The effects of phase differences between the AC voltage source and the parameter variation of inductance and capacitance has also been investigated.

Modification of the standard differential equations describing RC and RL circuits must be done with care since direct substitution of the inductance and capacitance terms (which are commonly considered as being constant) will result in an error. Additional errors are possible from the application of numerical methods for solving the differential equations. Specifically, to ensure numerical stability and accuracy of results the time step used for calculation should be significantly smaller than the time constants for the circuits being analysed.

Analytical solutions to RL and RC circuits involving cyclic variation in inductance and capacitance parameters are not widely available. Additionally, extension to represent circuits where inductance and capacitance are varied by switching and therefore change in discrete steps would not lend itself to analytical evaluation. Validation of the numerical methods used to arrive at solutions to the first order differential equations investigated in the present paper have been done in two ways: firstly, by comparison against analytical solutions for constant inductance/capacitance where the terms driving the parameter variation in the numerical model have been set to provide constant inductance/capacitance values with respect to time, and secondly by variation of the time step used for calculation which should result in minimal change to the numerically obtained solution providing significant errors are not present. For the validation by comparison against equivalent analytical solutions the time step has deliberately been set larger than the value used for the later calculations resulting in a somewhat coarse representation of the waveforms which was still found to produce acceptable results.

The analysis of the circuit problems has been done to the effect of calculating the average values for the apparent power in the circuit and the true power dissipated through the resistive load if the circuits were to be operated continuously and under steady state conditions. The ratio of these values has also been expressed as the commonly referred to power factor. True and apparent power has been calculated by numerical integration of the instantaneous power values calculated by numerically solving the differential equations for the current in the circuits as a function of time.

2. Derivations

2.1. Series RL Circuit with Time Varying Inductance

Letting V_S be the source voltage, V_R being the voltage across the series resistance R and V_L being the voltage across the series inductance L . Also, considering V_S and L to both be functions of time t . Using Kirchoff's voltage law:

$$V_S(t) = V_R(t) + V_L(t)$$

The magnetic flux ϕ is the product of the instantaneous inductance and current.

$$\phi = L(t)I(t)$$

V_L is equal to the rate of change in magnetic flux. Using the product rule to differentiate $L(t)I(t)$ gives,

$$V_L(t) = \frac{d\phi}{dt} = L(t) \frac{dI}{dt} + I(t) \frac{dL}{dt}$$

Which gives the first order differential equation where the solution is $I(t)$.

$$V_S(t) = I(t)R + L(t) \frac{dI}{dt} + I(t) \frac{dL}{dt}$$

Discretising using forwards differences and rearranging gives the current at the next iteration based on the current and inductance values at the previous iteration and the rate of change in inductance over the time step.

$$V_{S n} = I_n R + L_n \frac{I_{n+1} - I_n}{\Delta t} + I_n \frac{L_{n+1} - L_n}{\Delta t}$$

$$I_{n+1} = I_n + \frac{\Delta t}{L_n} \left(V_{S n} - I_n R - I_n \frac{L_{n+1} - L_n}{\Delta t} \right)$$

2.2. Series RC Circuit with Time Varying Capacitance

Applying the same process to a series RC circuit with capacitance C a function of time, driven by the time varying voltage source V_S .

$$V_S(t) = V_R(t) + V_C(t)$$

The charge q on the capacitor is equal to the instantaneous capacitance and voltage across the capacitor.

$$q = C(t)V_C(t)$$

Differentiating using the product rule gives the current $I(t)$ for a time varying capacitance which can be substituted back into Kirchoff's voltage law giving a first order differential equation with solution $V_C(t)$.

$$I(t) = \frac{dq}{dt} = C(t) \frac{dV_C}{dt} + V_C(t) \frac{dC}{dt}$$

$$V_S(t) = I(t)R + V_C(t) = R \left(C(t) \frac{dV_C}{dt} + V_C(t) \frac{dC}{dt} \right) + V_C(t)$$

$$V_S(t) = RC(t) \frac{dV_C}{dt} + RV_C(t) \frac{dC}{dt} + V_C(t)$$

Again, discretising using forwards differences and rearranging gives the capacitor voltage at the next iteration which can then be used to find the current by subtracting of the instantaneous capacitor voltage from the driving source voltage and dividing though by the series resistance.

$$V_{S n} = RC_n \frac{V_{C n+1} - V_{C n}}{\Delta t} + RV_{C n} \frac{C_{n+1} - C_n}{\Delta t} + V_{C n}$$

$$V_{C\,n+1} = V_{C\,n} + \frac{\Delta t}{RC_n} \left(V_{S\,n} - R V_{C\,n} \frac{C_{n+1} - C_n}{\Delta t} - V_{C\,n} \right)$$

3. Validations of Numerical Solutions

3.1. Comparison Against Analytical Solutions

3.1.1. RL Circuit

Parameter	Numerical	Analytical	Units
Time step	0.0001	-	s
V_max	200		V
Frequency	60		1/s
Load resistance	100		Ω
Inductance	0.01		H
Power avg period	1 to 6		s
Power factor	1.000711461	1.000000428	-
Apparent power	199.8585144	199.7167782	W
True power	200.0007059	199.7168636	W

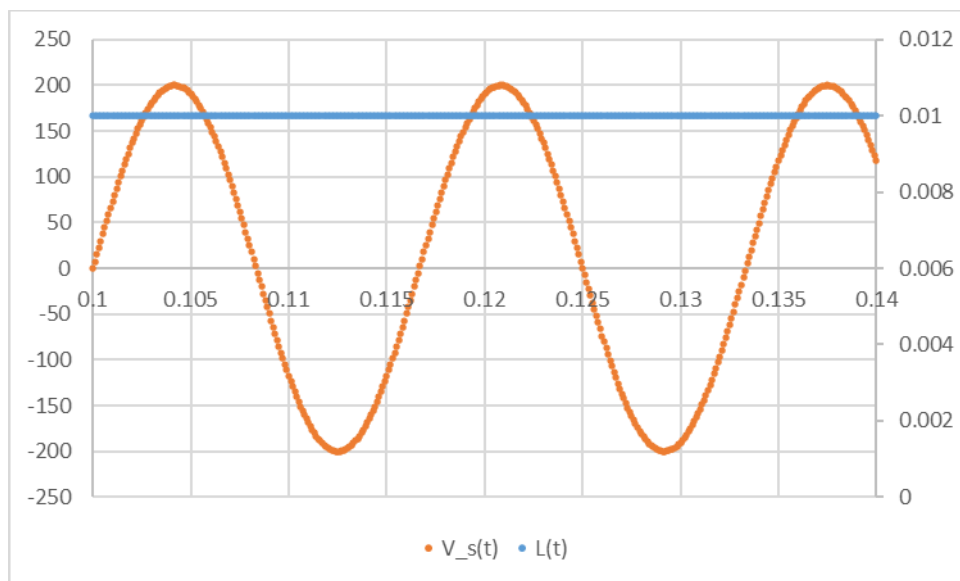


Figure 1 Source voltage function and inductance

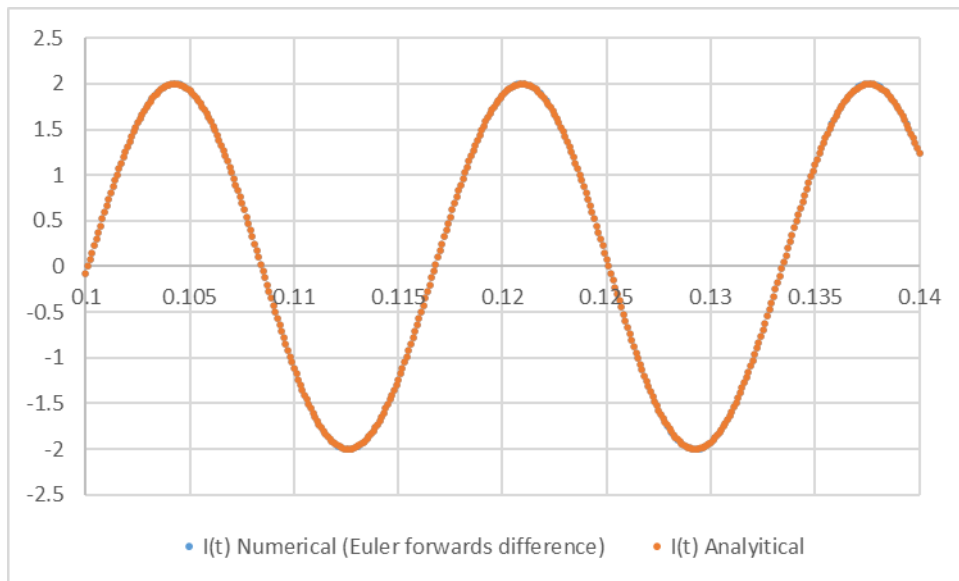


Figure 2 Numerical and analytical solutions for $I(t)$

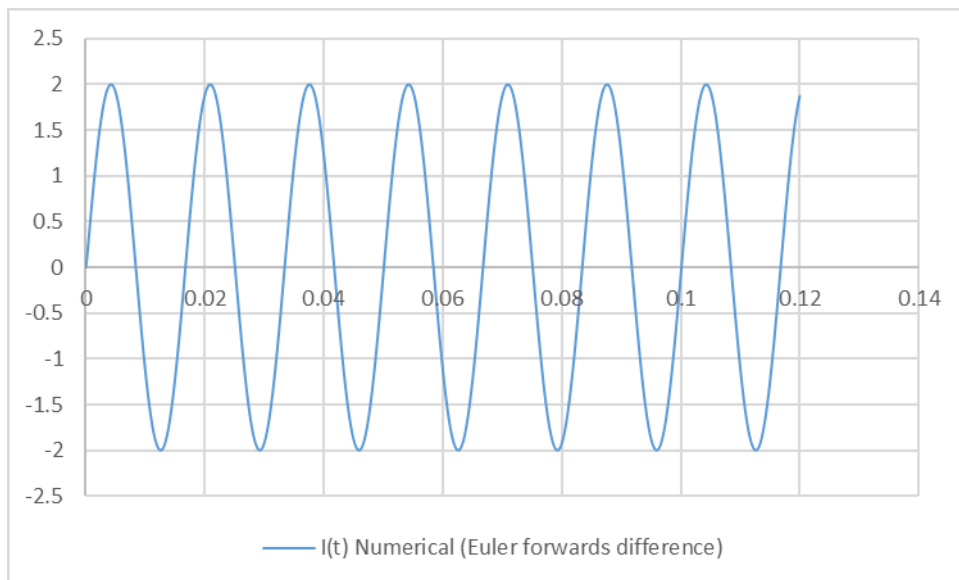


Figure 3 Transient period

3.1.2. RC Circuit

Parameter	Numerical	Analytical	Units
Time step	0.0001	-	s
V_max	200		V
Frequency	60		1/s
Load resistance	50		Ω
Capacitance	0.001		F
Power avg period	1 to 6		s
Power factor	1.001000566	0.9999999566	-
Apparent power	399.2755398	398.8782411	W
True power	399.6750413	398.8780678	W

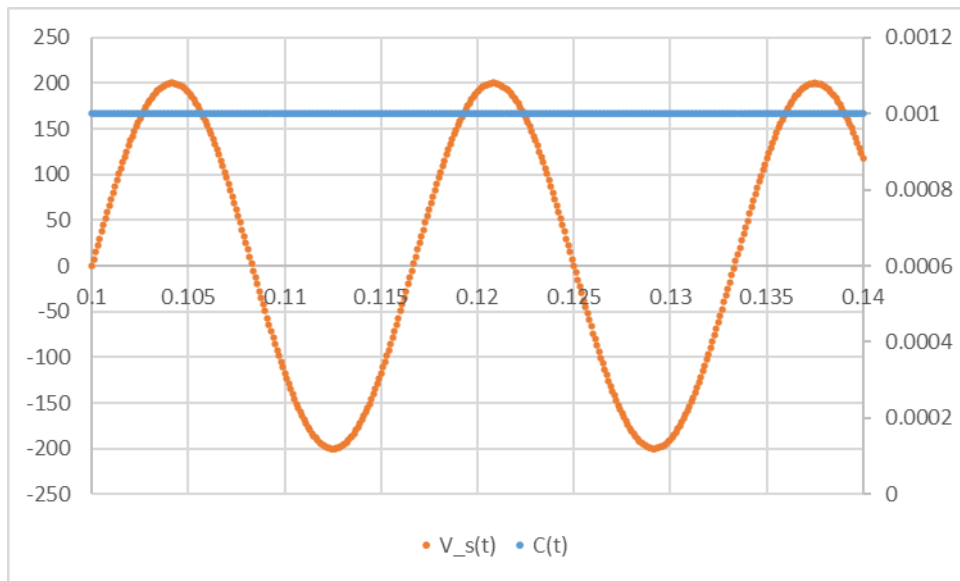


Figure 4 Source voltage function and capacitance

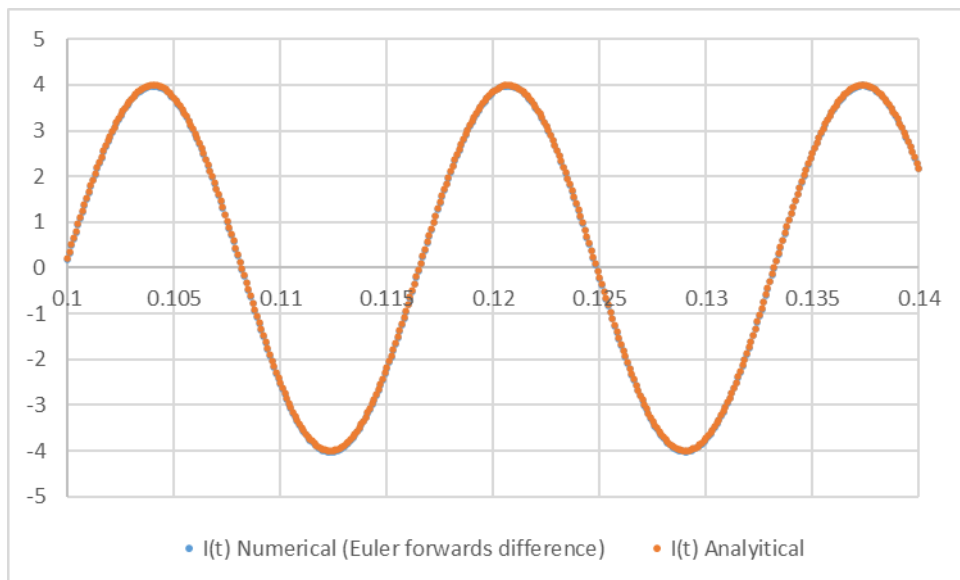


Figure 5 Numerical and analytical solutions for $I(t)$

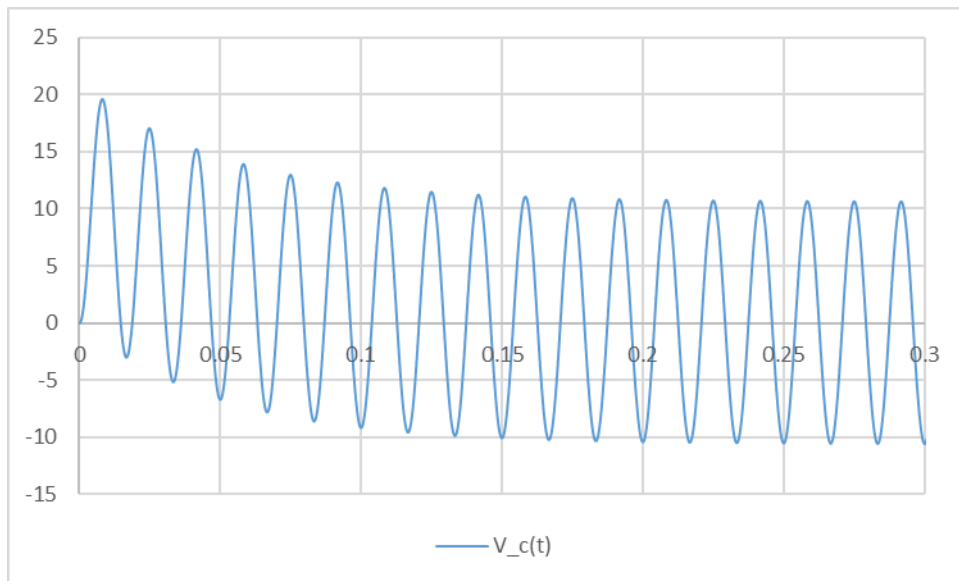


Figure 6 Transient period

4. Circuit Problems

4.1. Series RL Circuits with Time Varying Inductance

4.2. RL Circuit 1

Parameter	Value	Units
Time step	0.00002	S
V_max	200	V
Voltage source frequency	60	1/s
Load resistance	50	Ω
L_max	2	H
L_min	1	H
L variation frequency	120	1/s
L function phase angle	0 to 180	$^{\circ}$
Power avg period	0.2 to 1.2	s

Parameter	L function phase angle			Units
	0 $^{\circ}$	90 $^{\circ}$	180 $^{\circ}$	
Power factor	0.38994785	0.771959011	-1.181426993	-
Apparent power	9.707400861	3.204362136	-3.058035777	W
True power	3.785380095	2.473636225	3.612846012	W

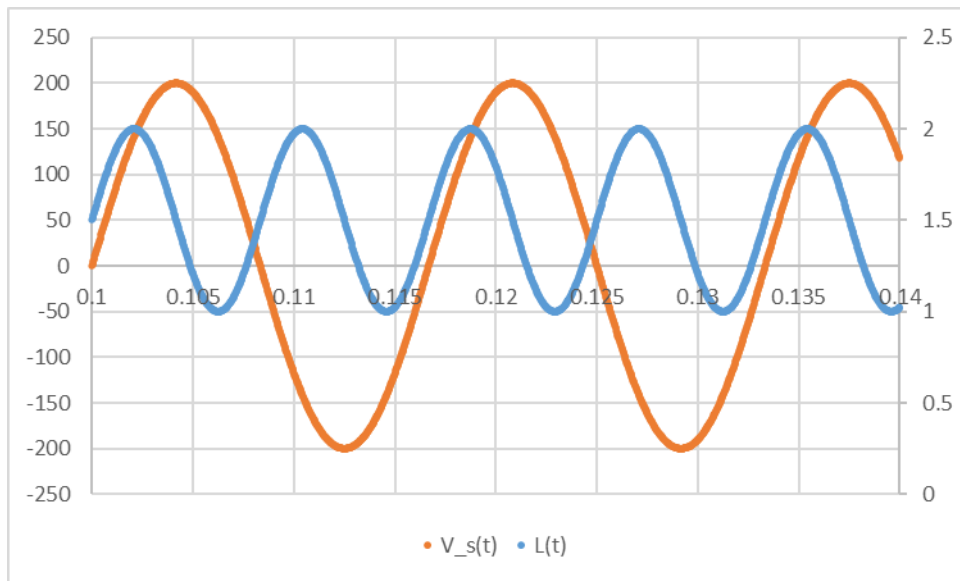


Figure 7 Source voltage function and inductance with 0° between inductance variations and source voltage

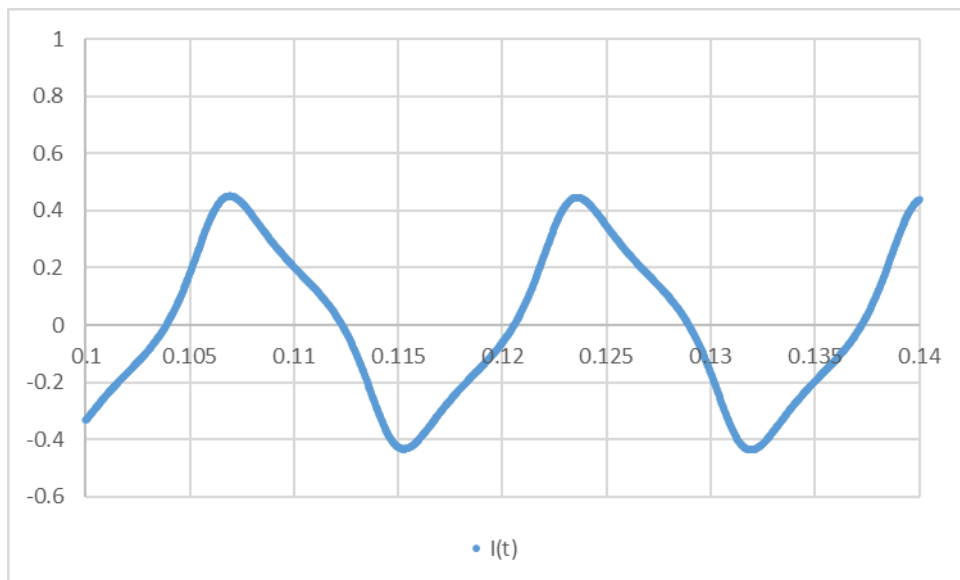


Figure 8 Current function with 0° between inductance variations and source voltage

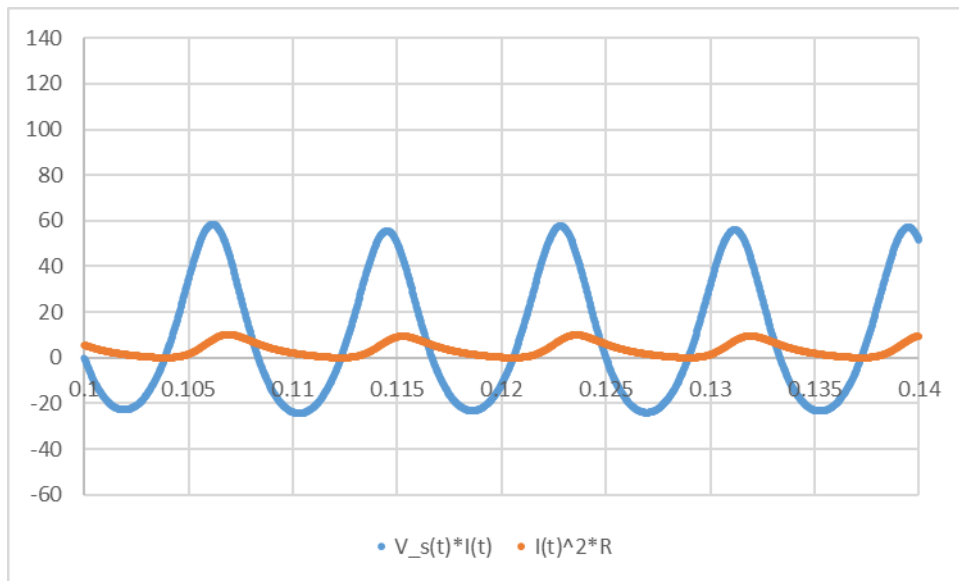


Figure 9 True and apparent instantaneous power vs time with 0° between inductance variations and source voltage

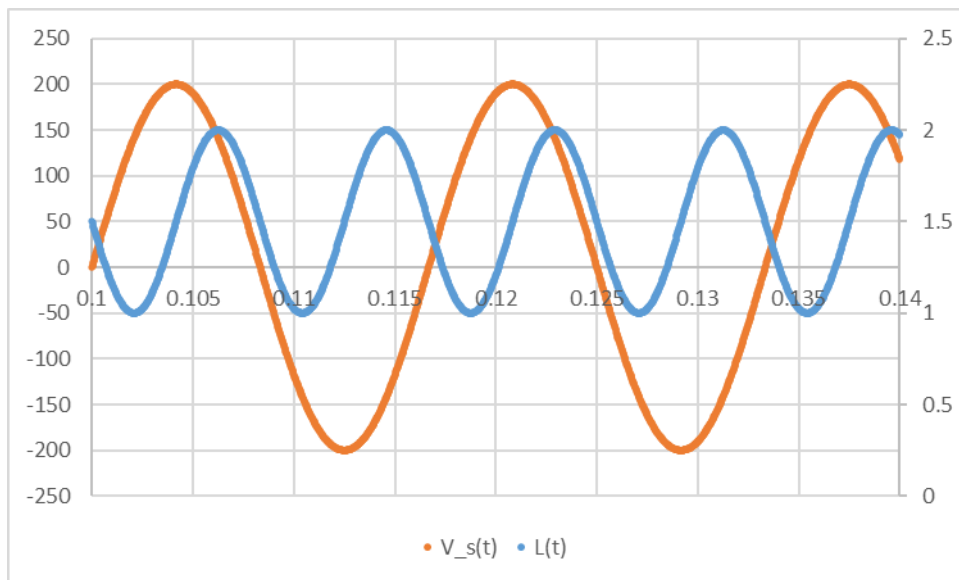


Figure 10 Source voltage function and inductance with 180° between inductance variations and source voltage

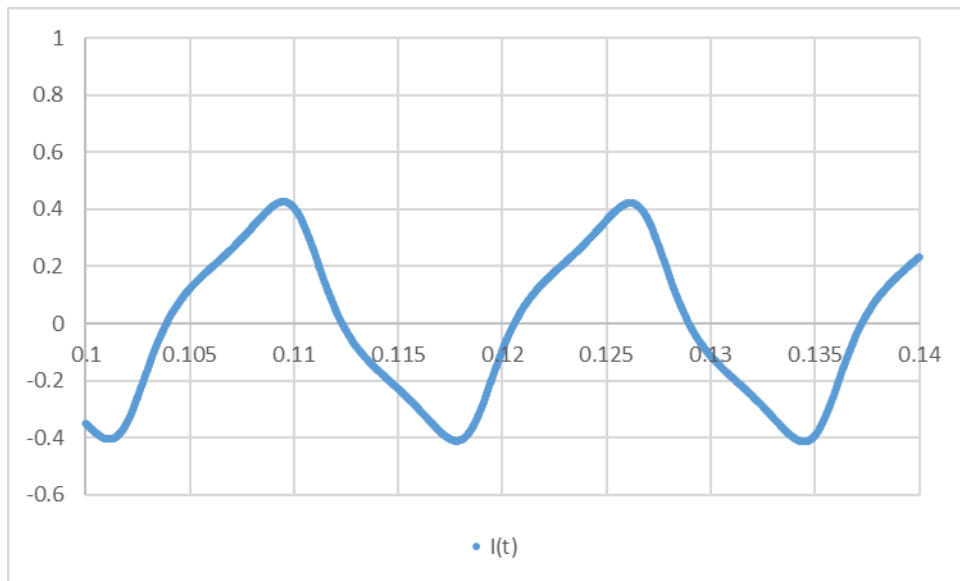


Figure 11 Current function with 180° between inductance variations and source voltage

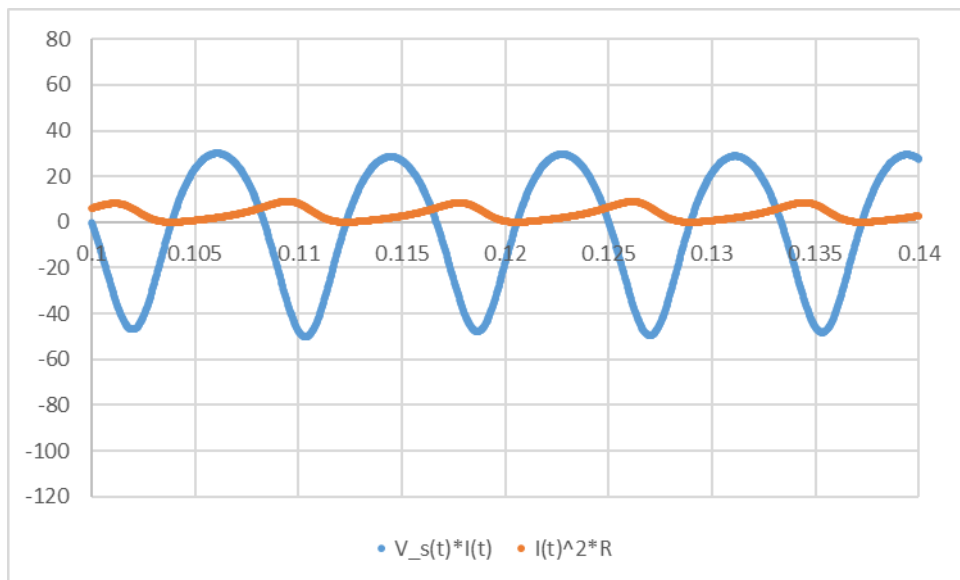


Figure 12 True and apparent instantaneous power vs time with 180° between inductance variations and source voltage

4.2.1. RL Circuit 2

Parameter	Value	Units
Time step	0.00002	S
V_max	200	V
Voltage source frequency	60	1/s
Load resistance	50	Ω
L_max	2	H
L_min	1	H
L variation frequency	60	1/s
L function phase angle	0 to 180	$^{\circ}$
Power avg period	0.2 to 1.2	s

Parameter	L function phase angle			Units
	0 $^{\circ}$	90 $^{\circ}$	180 $^{\circ}$	
Power factor	1.059397702	1.049740968	1.059200272	-
Apparent power	3.19736836	3.197035047	3.197956721	W
True power	3.387284694	3.356058665	3.38727663	W

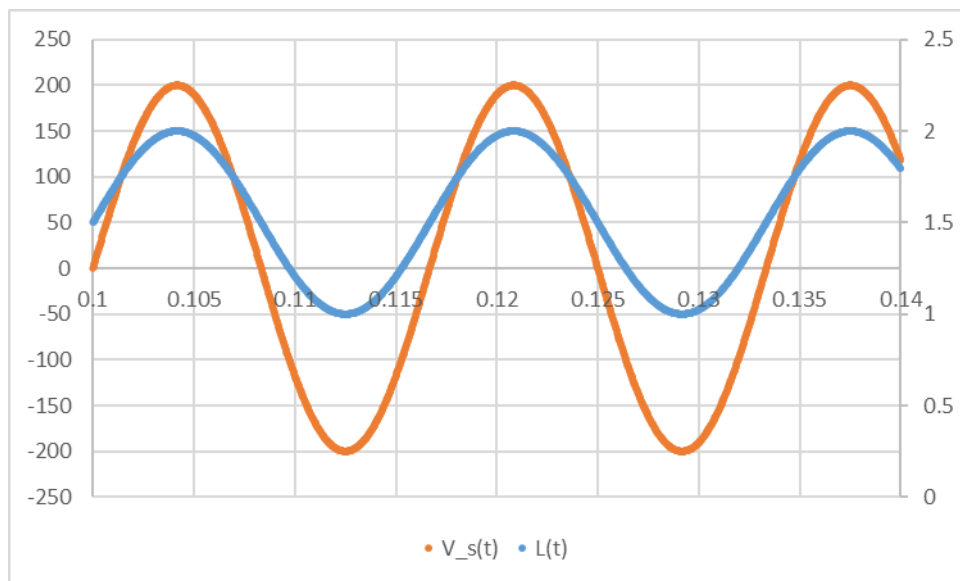


Figure 13 Source voltage function and inductance with 0° between inductance variations and source voltage

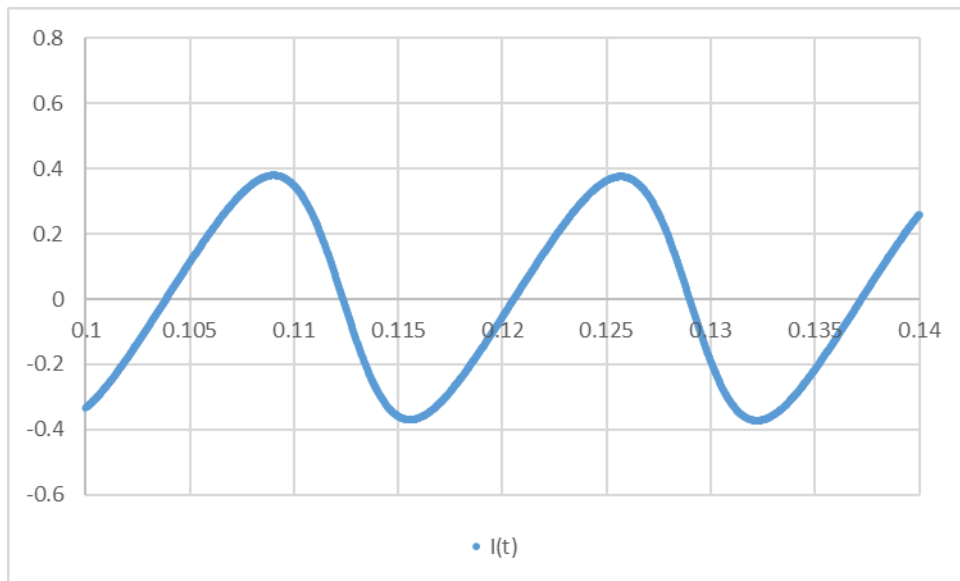


Figure 14 Current function with 0° between inductance variations and source voltage

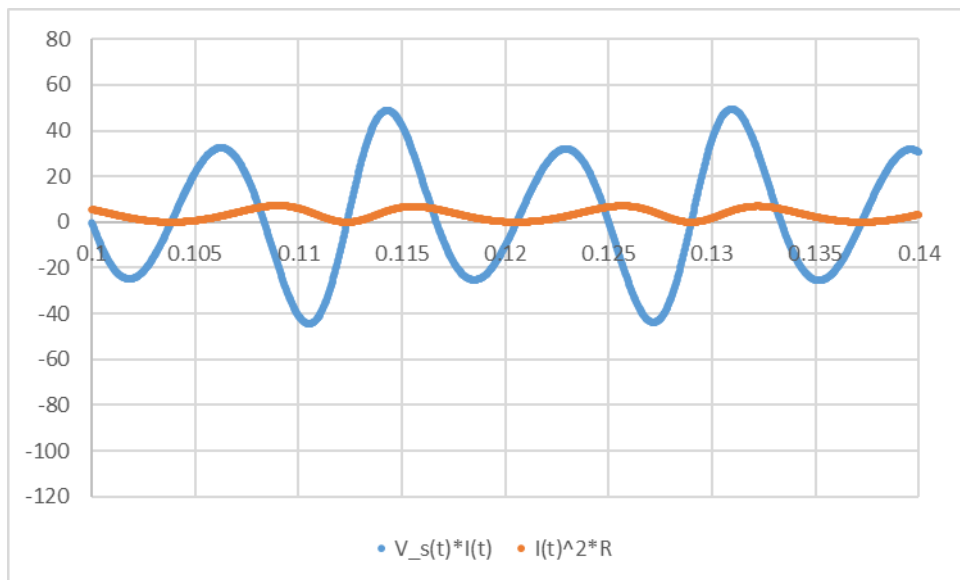


Figure 15 True and apparent instantaneous power vs time with 0° between inductance variations and source voltage

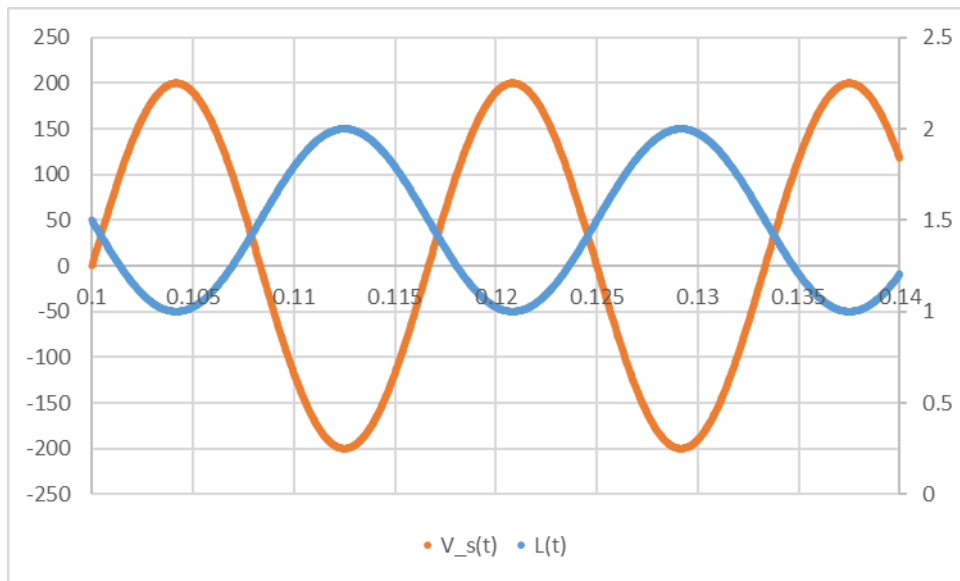


Figure 16 Source voltage function and inductance with 180° between inductance variations and source voltage

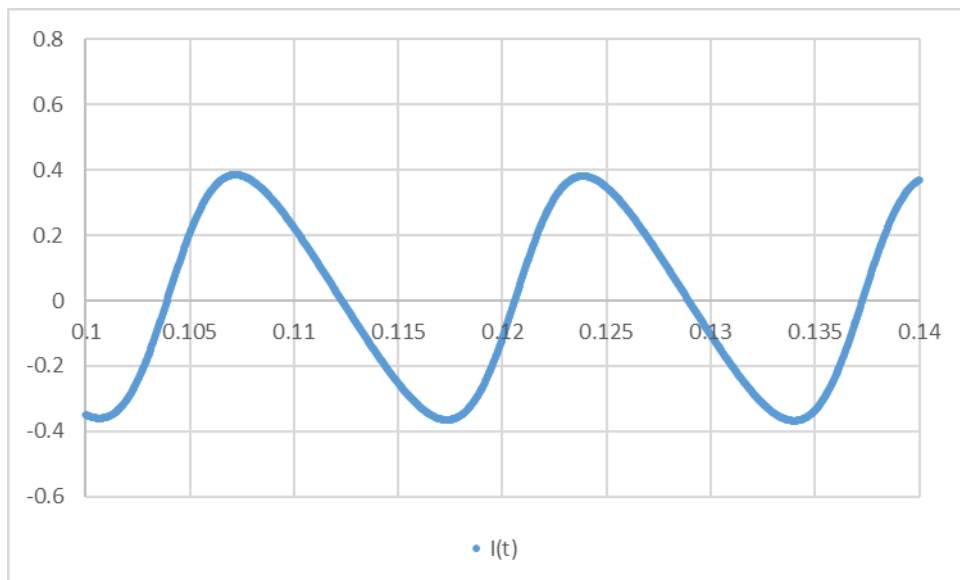


Figure 17 Current function with 180° between inductance variations and source voltage

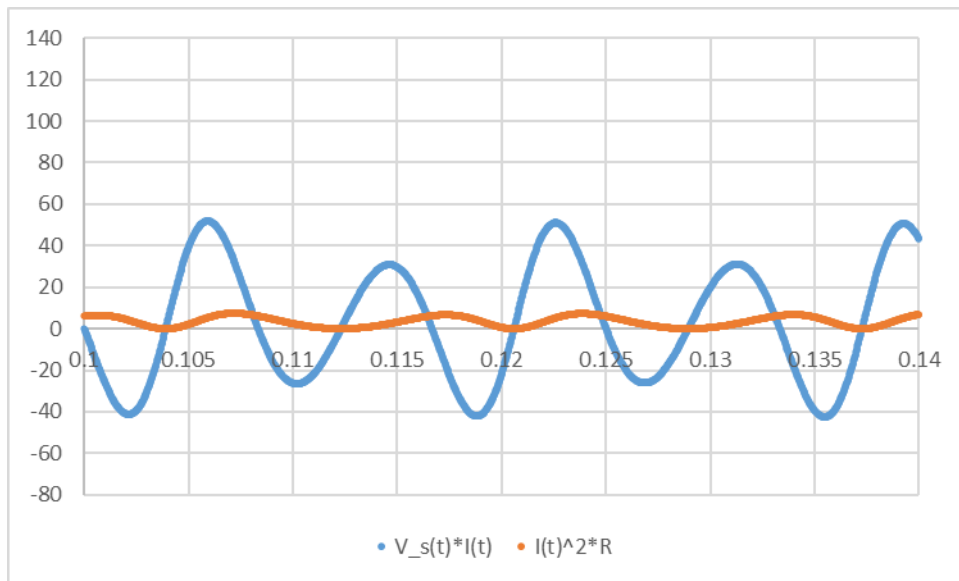


Figure 18 True and apparent instantaneous power vs time with 180° between inductance variations and source voltage

4.3. Series RC Circuit with Time Varying Capacitance

4.3.1. RC Circuit 1

Parameter	Value	Units
Time step	0.00001	S
V_max	200	V
Voltage source frequency	60	1/s
Load resistance	100	Ω
C_max	0.00001	H
C_min	0.000001	H
L variation frequency	120	1/s
L function phase angle	0 to 180	$^\circ$
Power Avg Period	0.1 to 0.6	s

Parameter	C function phase angle			Units
	0°	90°	180°	
Power Factor	-2.63930557	1.281096598	0.723324754	-
Apparent Power	-6.856017619	9.23854805	25.03255925	W
True Power	18.09512549	11.83547248	18.10666977	W

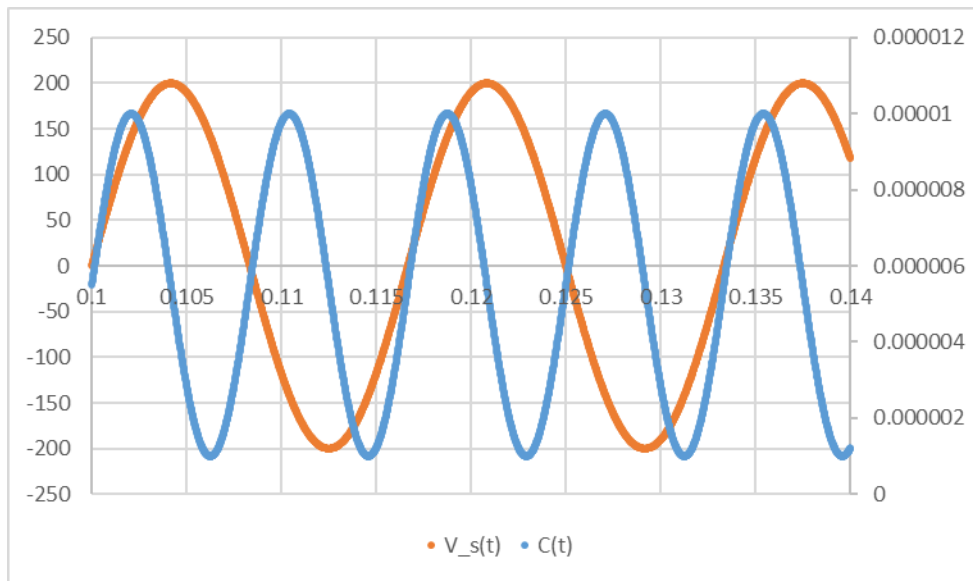


Figure 19 Source voltage function and capacitance with 0° between capacitance variations and source voltage

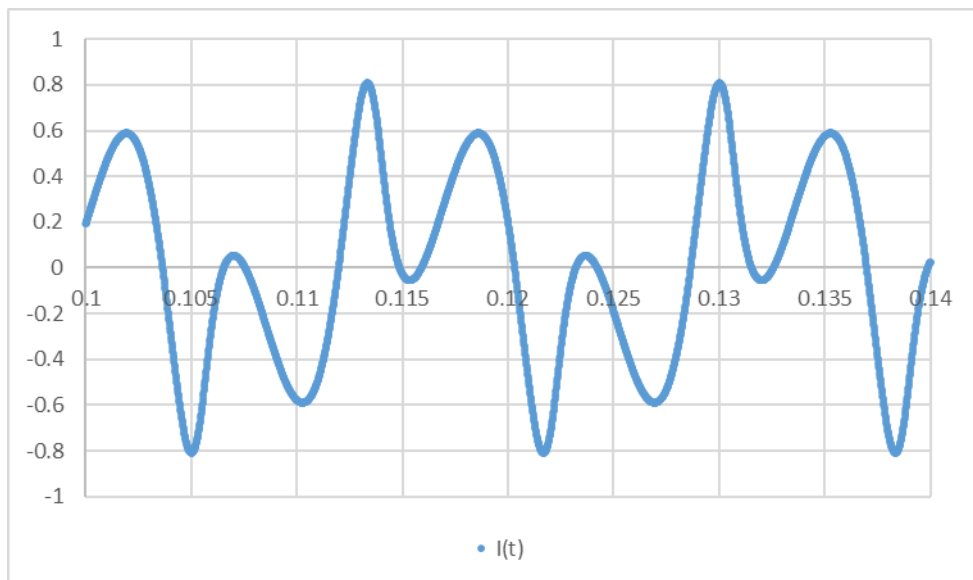


Figure 20 Current function with 0° between capacitance variations and source voltage

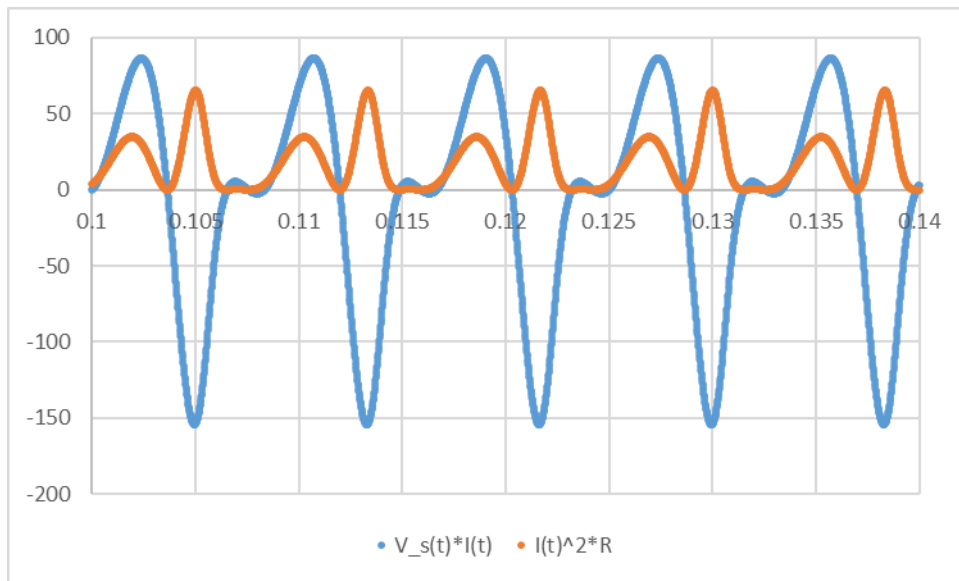


Figure 21 True and apparent instantaneous power vs time with 0° between capacitance variations and source voltage

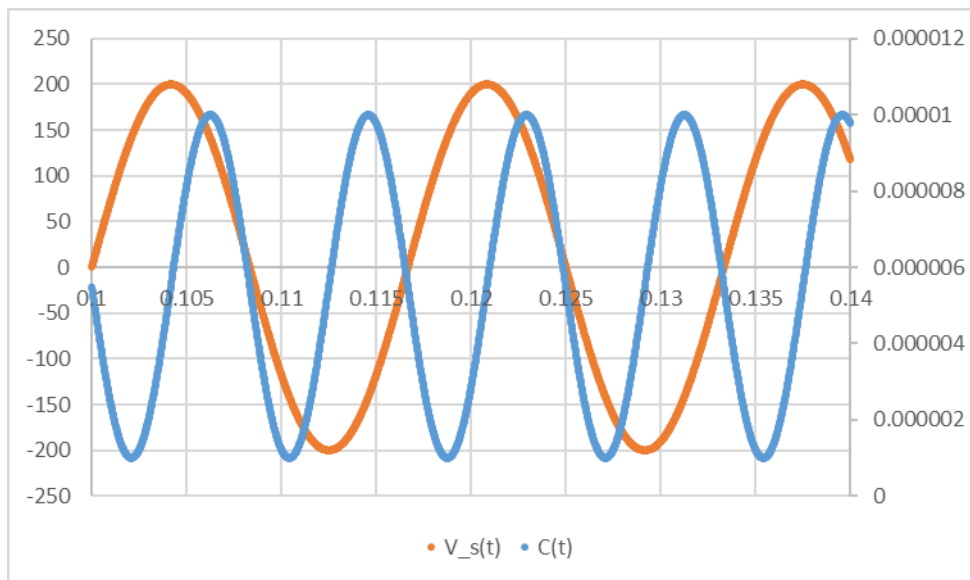


Figure 22 Source voltage function and capacitance with 180° between capacitance variations and source voltage

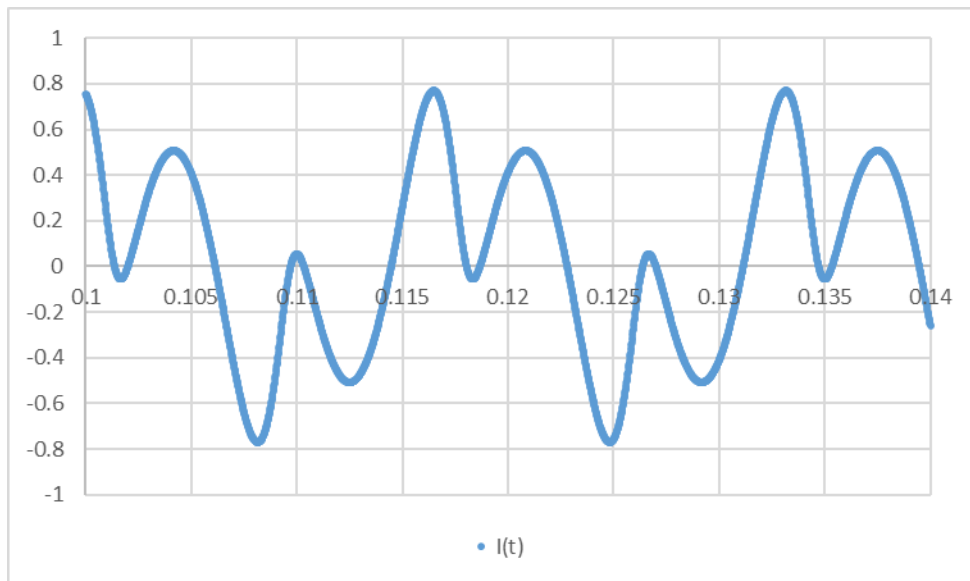


Figure 23 Current function with 180° between capacitance variations and source voltage

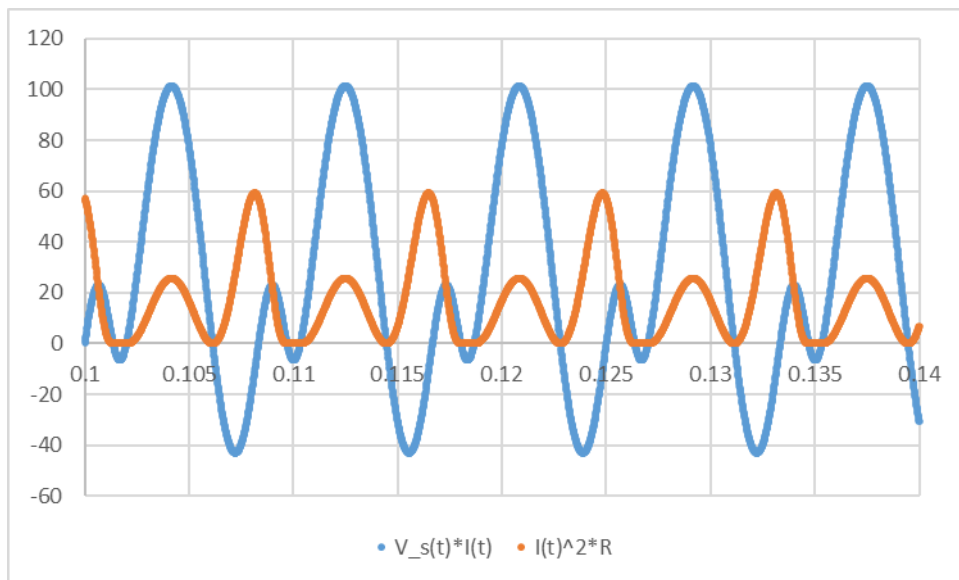


Figure 24 True and apparent instantaneous power vs time with 180° between capacitance variations and source voltage

4.3.2. RC Circuit 2

Parameter	Value	Units
Time step	0.00001	S
V_max	200	V
Voltage source frequency	60	1/s
Load resistance	100	Ω
C_max	0.00001	H
C_min	0.000001	H
L variation frequency	60	1/s
L function phase angle	0 to 180	$^{\circ}$
Power Avg Period	0.1 to 0.6	s

Parameter	C function phase angle			Units
	0 $^{\circ}$	90 $^{\circ}$	180 $^{\circ}$	
Power Factor	1.243611411	1.243263333	1.243682552	-
Apparent Power	9.971526621	9.971512142	9.971486862	W
True Power	12.40070429	12.39721542	12.40136423	W

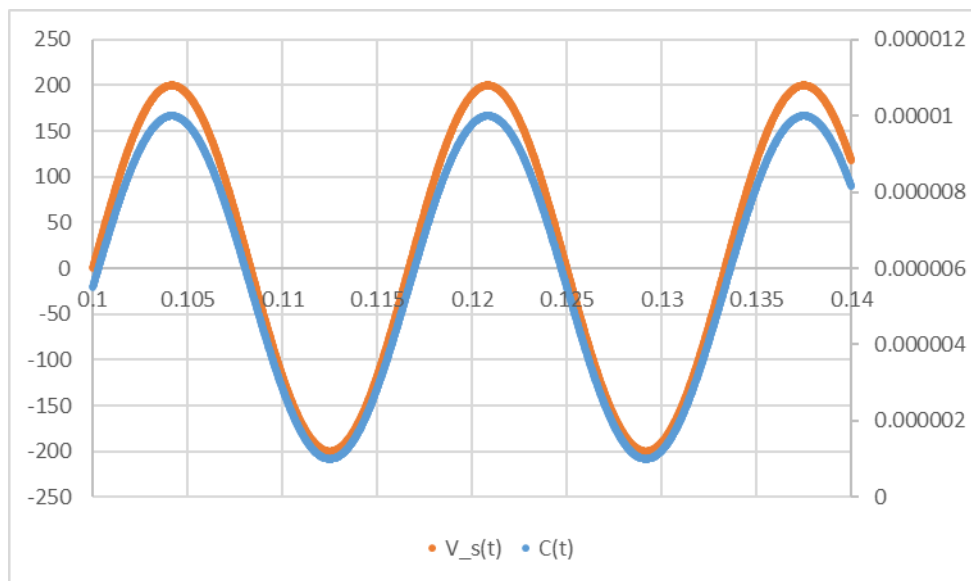


Figure 25 Source voltage function and capacitance with 0 $^{\circ}$ between capacitance variations and source voltage

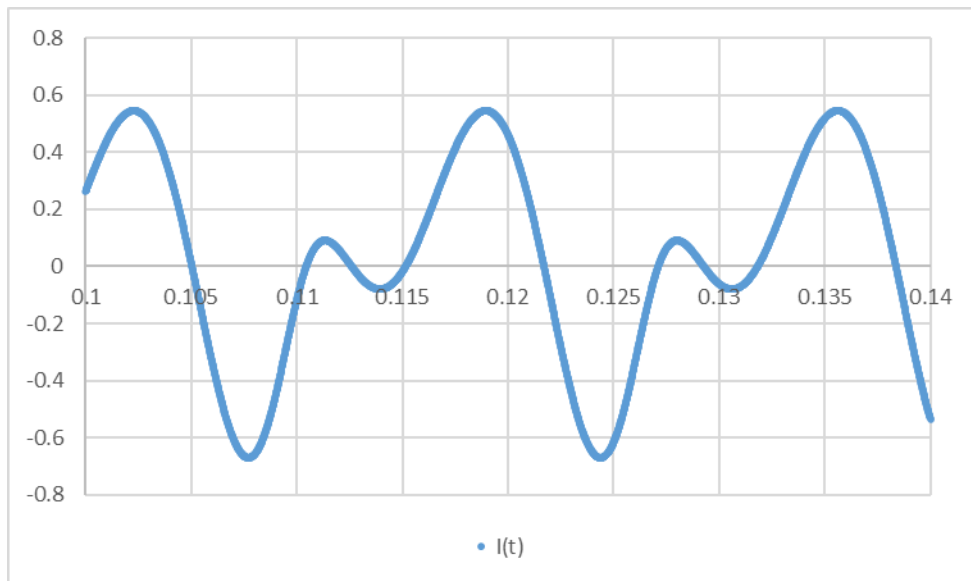


Figure 26 Current function with 0° between capacitance variations and source voltage

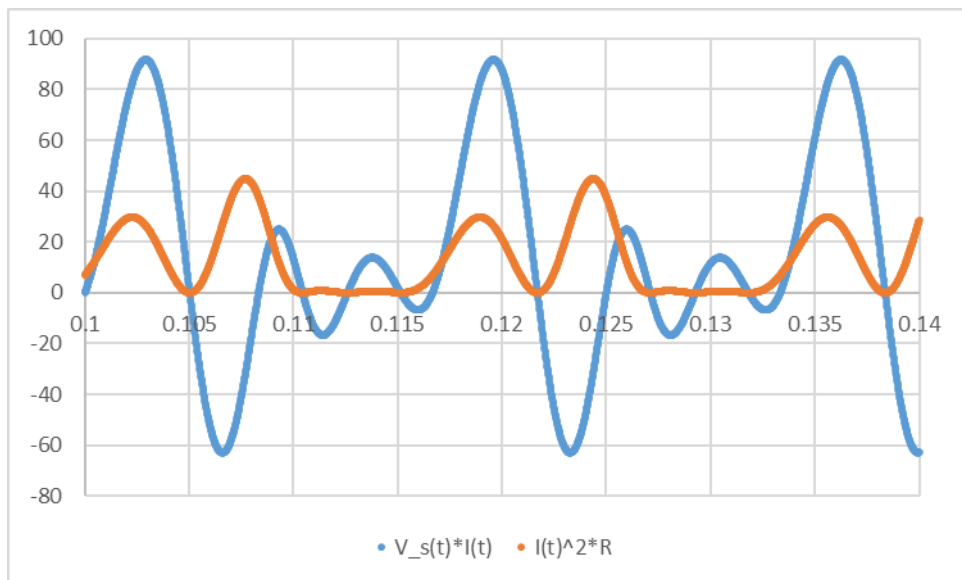


Figure 27 True and apparent instantaneous power vs time with 0° between capacitance variations and source voltage

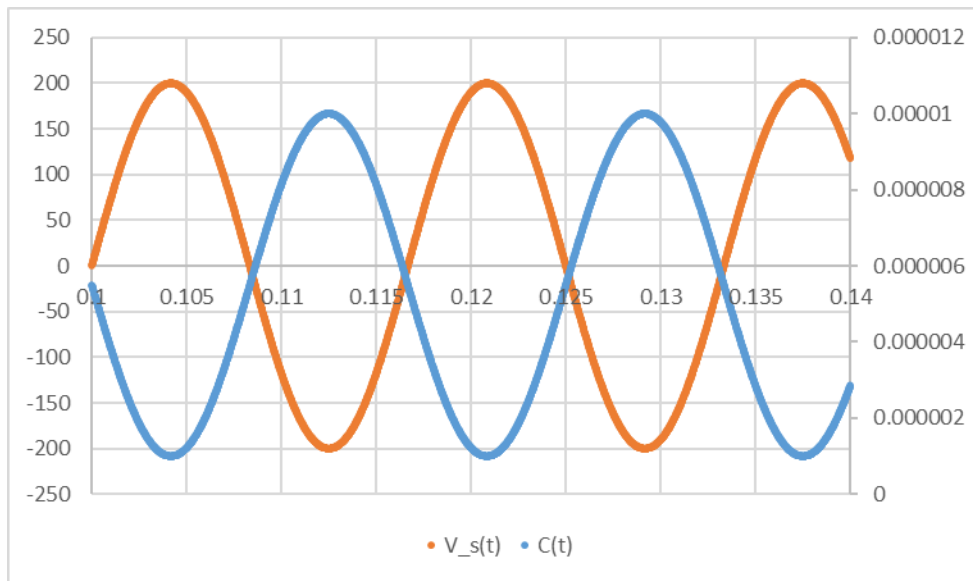


Figure 28 Source voltage function and capacitance with 180° between capacitance variations and source voltage

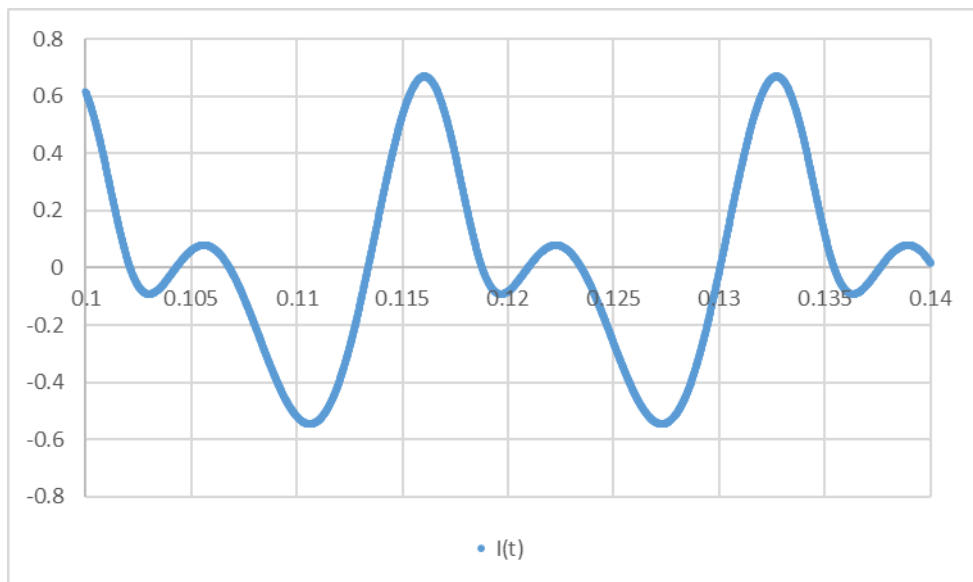


Figure 29 Current function with 180° between capacitance variations and source voltage

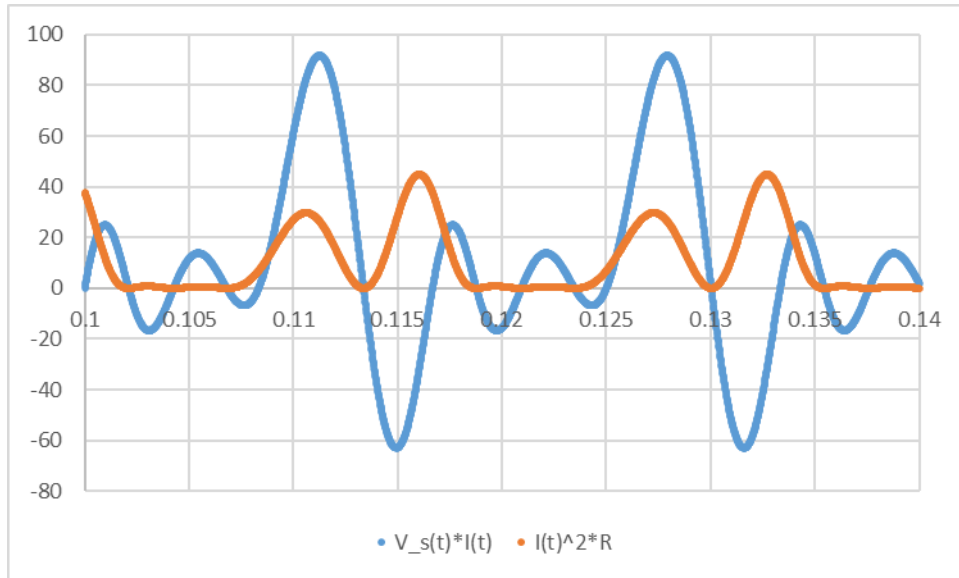


Figure 30 True and apparent instantaneous power vs time with 180° between capacitance variations and source voltage

4.4. Numerical Errors

4.4.1. RL Circuit 1

Parameter	Step size relative to Δt used for calculations			Units
	$1/2\Delta t$	Δt	$2\Delta t$	
Power Factor	-1.199810068	-1.181426993	-1.154440573	-
Apparent Power	-2.99879325	-3.058035777	-3.154046201	W
True Power	3.597982334	3.612846012	3.641158904	W

180° between L and V

4.4.2. RL Circuit 2

Parameter	Step size relative to Δt used for calculations			Units
	$1/2\Delta t$	Δt	$2\Delta t$	
Power Factor	1.026635303	1.059200272	1.107172772	-
Apparent Power	3.29809569	3.197956721	3.061449013	W
True Power	3.385941466	3.38727663	3.38955299	W

180° between L and V

4.4.3. RC Circuit 1

Parameter	Step size relative to Δt used for calculations			Units
	$1/2\Delta t$	Δt	$2\Delta t$	
Power Factor	-2.638458752	-2.63930557	-2.641099665	-
Apparent Power	-6.856287465	-6.856017619	-6.855354082	W
True Power	18.09003167	18.09512549	18.10567337	W

0° between C and V

4.4.4. RC Circuit 2

Parameter	Step size relative to Δt used for calculations			Units
	$1/2\Delta t$	Δt	$2\Delta t$	
Power Factor	1.23926877	1.243263333	1.252387621	-
Apparent Power	9.994618015	9.971512142	9.925219066	W
True Power	12.38601797	12.39721542	12.43022149	W

0° between C and V

5. Discussion

When analysed through the previously described means both the RC and RL circuits exhibited behaviour where more power on average was absorbed by the load by comparison to the power in the circuit. In both the RC and RL circuits when the parameter variation of inductance or capacitance with respect to time was at a frequency double that of the driving voltage source the power factor or ratio of true to apparent power was found to depend very strongly on the phase difference between the parameter variation and voltage source (see Figure 31).

Conversely, when the parameter variation of inductance and capacitance was set to a frequency equal to the driving AC voltage source frequency the phase difference was more or less independent of the calculated ratios of true to apparent power. It is worth noting that in the case of the RC circuit even in this condition the average power absorbed by the load was still calculated to be in excess of the apparent power in the circuit by a value greater than what could be reasonably explained by numerical error alone.

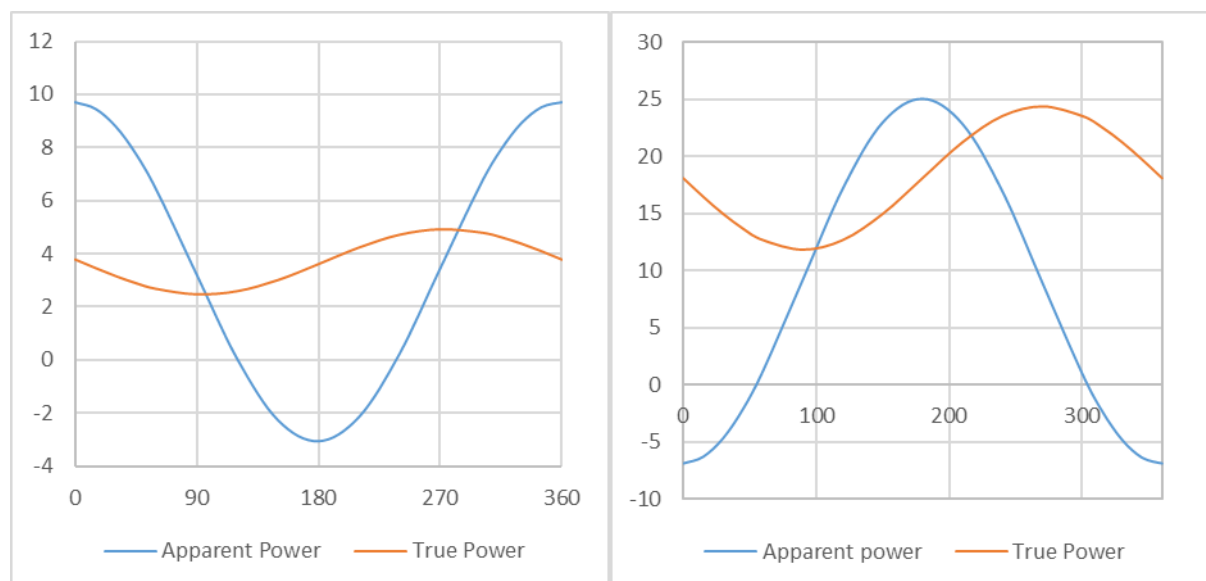


Figure 31 Apparent power and true power vs phase angle between parameter variation and voltage source (left - RL circuit, right RC circuit) both with parameter variation at double the voltage source frequency

In the RL circuit cases under circumstances of producing variation of inductance mechanically through use of a device such as a variometer inductor, the excess power delivered to the load may be accounted for as the power required to overcome magnetic resistance due to Lenz's law. In this case the torque required to turn the variometer is equal to the rate of change of energy stored within the magnetic field of the inductor. The energy stored in the magnetic field is given by,

$$W = \frac{1}{2}L(\theta)I(t)^2$$

Where θ is the angle between the two coils of the variometer. Differentiation with respect to θ gives the torque τ as,

$$\tau = \frac{1}{2}I(t)^2 \frac{dL}{d\theta}$$

Considering the first RL circuit with the parameter variation and inductance 180° out of phase with the voltage source and double the frequency, the calculated true power absorbed by the load was 3.612846012W vs -3.058035777W apparent power in the circuit where the negative indicates more power being returned from the load vs the power being consumed by the load. Calculation of the power required to turn the variometer by numerical integration of the power required at each time step found using the previous formula yields 6.607589976W giving a corrected power factor of 1.017830919 very close to unity explaining the energy discrepancy.

A similar calculation could be made on forces exerted on dielectric materials in devices which can vary their total capacitance mechanically, which would likely also bring the calculated power factors from the RC circuit problems back to unity. However, the calculation of the behaviour of devices where inductance and capacitance can be varied by switching capacitors or inductors through series and parallel configuration, or by electronically varying the length of the active portion of a coil providing inductance cyclically with respect to time presents challenges for numerical analysis owing to the rapid discrete changes in circuit parameters. It would be possible to approximate a sinusoidal variation of inductance or capacitance in a series of discrete steps by a complex switching process changing a bank of inductors or capacitors between different combinations of series and parallel configurations, in this case it would be reasonable to assume similar behaviour compared with analysis performed with the assumption of a continuously varying function with the additional energy involved in creating the parameter variation coming from the power circuitry required to control the switches, which would not seem to be directly related to the parameter variation itself.